DE

i) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 4y = 0$ — ordinary DE

since only one independent variabl

ii) $\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = 0$ — partial DE

since 2 independent variables.

i) <u>order</u> :- The highest derivative involved in the DE is called order of DE.

(ii) <u>Degree</u> :- The highest power of highest derivative is called degree of DE (provided the derivatives of the dependent variable should free from radicals & fractions).

① $\left(\frac{d^2y}{dx^2}\right)^2 = \left[x + 4\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

$$\left(\frac{d^2y}{dx^2}\right)^4 = \left[x + 4\left(\frac{dy}{dx}\right)^2\right]^3$$

order — 2   degree — 4

② $\frac{dy}{dx} = x + \frac{2}{\frac{dy}{dx}}$

$$\left(\frac{dy}{dx}\right)^2 = x\left(\frac{dy}{dx}\right) + 2$$

order = 1
degree = 2.

③ $dr = (\theta + \cos\theta)\, d\theta$

$$\frac{dr}{d\theta} = \theta + \cos\theta$$

order = 1
degree = 1.

④ $\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 + 3y = 0$ order = 3
degree = 1.

⑤ $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial y^2}\right)^4$

order = 2

⇒ Formation of ordinary DE :-

$$F(x, y, a, b) = 0 \quad —① $$

diff ① w.r.t x.

$$F_1\left(x, y, \frac{dy}{dx}, a, b\right) = 0 \quad —②$$

diff ② w.r.t x

$$F_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b\right) = 0 \quad —③$$

eliminating a & b from above 3 eq

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

The no. of arbitrary constants eliminated should be equal to the order of resulting ordinary DE.

* The DE of family of straight line passing through the origin is

(a) $y\,dy + x\,dx = 0$ b) none

~~c)~~ $y\,dx - x\,dy = 0$ d) $y\,dy - x\,dx = 0$

$$y = mx$$

$$\frac{dy}{dx} = m \quad \Rightarrow \quad y = \frac{dy}{dx} \cdot x$$

$$y\,dx - x\,dy = 0$$

⊛ $y = c(x-c)^2$ c is arbitrary const

$$y' = \frac{dy}{dx} = 2c(x-c)$$

$$\frac{y}{y'} = \frac{x-c}{2} \Rightarrow x - c = \frac{2y}{y'}$$

$$c = x - \frac{2y}{y'}$$

DE of family of circles

Note : If the given eqⁿ of the form $y = Af(x) + Bg(x)$ then the resulting DE is

$$\begin{vmatrix} y & f & g \\ y' & f' & g' \\ y'' & f'' & g'' \end{vmatrix} = 0.$$

$y = Ae^{2x} + Bx.$

$$\begin{vmatrix} y & e^{2x} & x \\ y' & 2e^{2x} & 1 \\ y'' & 4e^{2x} & 0 \end{vmatrix} = 0$$

$y(-4) - 1(0 - y'') + x(4y' - 2y'') = 0$

$y''(1 - 2x) + 4xy' - 4y = 0.$

$y = e^x (A\cos x + B\sin x)$ —— ①

$y' = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$

$y' = e^x(-A\sin x + B\cos x) + y.$

$y'' = y' + e^x(-A\cos x - B\sin x) +$
$e^x(-A\sin x + B\cos x).$

$y'' = y' - y + y' - y$

$y'' - 2y' + 2y = 0$

Note :- If the given eq of the form $y = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx} + \dots$ then

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es.

$$\boxed{\begin{gathered}(D-a)(D-b)(D-c)\,y = 0 \\ D = \frac{d}{dx} \qquad D^2 = \frac{d^2}{dx^2}\end{gathered}}$$

$$y = Ae^{2x} + Be^{-3x}$$

$$(D-2)(D+3)y = 0$$

$$(D^2 + D - 6)y = 0$$

$$y'' + y' - 6y = 0$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$$

$$(D+1)(D+2)(D+3)\,y = 0$$

$$(D^2 + 3D + 2)(D+3)\,y = 0$$

$$(\underset{\substack{\downarrow \\ 1+2+3}}{D^3 + 6D^2} + \underset{\substack{\downarrow \\ 1\cdot2+2\cdot3+3\cdot1}}{11D} + \underset{\substack{\downarrow \\ 1\cdot2\cdot3}}{6})\,y = 0$$

$$y''' + 6y'' + 11y' + 6y = 0$$

> Solution of DE :-

$$\frac{d^2y}{dt^2} = g \qquad y(0) = 0 \quad y'(0) = 0$$

$$\frac{dy}{dt} = gt + c_1$$

$$y = \frac{1}{2}gt^2 + c_1 t + c_2$$

$y(0) = 0 \Rightarrow c_2 = 0$

$y'(0) = 0 \Rightarrow c_1 = 0$

$$y = \frac{1}{2}gt^2$$

$\Rightarrow$ 1st order 1st degree DE :-

$$\frac{dy}{dx} = F(x, y)$$

(or)

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Variable - seperable method :-

* $$\frac{dy}{dx} = e^{x-y} + x^3 e^{-y}$$

$$= e^{-y}(e^x + x^3)$$

$$\int e^y\,dy = \int (e^x + x^3)\,dx$$

$$e^y = e^x + \frac{x^4}{4} + c$$

* $$\log\left(\frac{dy}{dx}\right) = 2x + 3y$$

$$\frac{dy}{dx} = e^{2x} \cdot e^{3y}$$

$$\int e^{-3y}\,dy = \int e^{2x}\,dx$$

$$\frac{-1}{3} e^{-3y} = \frac{e^{2x}}{2} + c$$

* $$\frac{dy}{dx} = \frac{y^2}{1-xy}$$

$$(1-xy)\,dy = y^2 dx$$

$$dy = y^2 dx + xy\,dy$$

$$dy = y(y\,dx + x\,dy)$$

$$\int \frac{1}{y}\,dy = \int d(xy)$$

$$\log y = xy + c$$

* $$\frac{dy}{dx} = \frac{-x}{y}$$ at $x = 1$, $y = \sqrt{3}$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + c \Rightarrow c = \frac{3}{2} + \frac{1}{2} = 2$$

$$y^2 = -x^2 + 4$$

* $$\frac{dy}{dx} = y^2 \sin x \;;\; y(2\pi) = 1$$

a) $y \sin x = 1$ $\quad$ ~~b)~~ $y \cos x = 1$

$$\int \frac{1}{y^2}\, dy = \int dx\, \sin x$$

$$\frac{-1}{y} = -\cos x + c \quad \Rightarrow c = 0$$

$$y \cos x = 1 .$$

$\frac{dy}{dx} = 3x^2 - 2x$ passes through $(1,1)$ then find mag of y when $x = 3$

$$\int dy = \int 3x^2 - 2x\, dx .$$

$$y = x^3 - x^2 + c .$$

$(1,1) \Rightarrow c = 1 .$

$$y = x^3 - x^2 + 1 .$$

when $x = 3 \Rightarrow y = 19 .$

Bio transformation of an organic compound having conc. 'x' can be modelled using DE $\frac{dx}{dt} + kx^2 = 0$. At $t = 0$ the conc is 'a' then the soln is

$$\frac{dx}{dt} = -kx^2$$

$$\int \frac{1}{x^2}\, dx = -k \int dt$$

$$\frac{-1}{x} = -kt + c \quad \Rightarrow c = -1/a .$$

$$\frac{dx(t)}{dt} + 3x(t) = 0 .$$

$$\int \frac{1}{x}\, dx = \int -3\, dt .$$

$$\log x = -3t + \log c .$$

$$x = e^{-3t} c$$

$$\frac{dy}{dx} = 1 + y^2$$

$$\int \frac{1}{\quad}\, dy = \int dx$$

$$\tan^{-1} y = x + c$$

$$y = \tan(x+c).$$

⊛ Find the curve passing through the point $(0, 1)$ & satisfying $\sin\left(\frac{dy}{dx}\right) = b$.

$$\frac{dy}{dx} = \sin^{-1} b$$

$$\int dy = \int \sin^{-1} b \, dx$$

$$y = (\sin^{-1} b)x + c$$

$(0,1)$ $\quad c = 1$

$$y = (\sin^{-1} b)x + 1.$$

⊛ $\frac{dy}{dx} = e^{x+y}$ given that for $x = 1, y = 1$; find y when $x = -1$.

$$\int e^{-y} \, dy = \int e^{x} \, dx$$

$$-e^{-y} = e^{x} + c \quad \Rightarrow \quad c = -e^{-1} - e.$$

$$\Rightarrow \; -e^{-y} = e^{x} - (e^{-1} + e)$$

at $x = 1$ $\Rightarrow$ $-e^{-y} = \cancel{e^{-1}} - \cancel{e^{1}} - \cancel{e}\, e$

$$y = -1.$$

⊛ $\frac{dy}{dx} = (4x + y + 1)^2$

$4x + y + 1 = v.$

$4 + \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{dv}{dx} - 4.$

$$\frac{dv}{dx} = v^2 + 4$$

$$\int \frac{1}{v^2 + 2^2} \, dv = \int dx$$

$$\frac{1}{2} \tan^{-1}\left(\frac{v}{2}\right) = x + c.$$

$$\frac{4x + y + 1}{2} = \tan(2x + c).$$

$$4x + y + 1 = 2\tan(2x + c).$$

omogenous differential eq^n :-

$$\frac{dy}{dx} = F(x,y)$$

is said to be homogenous DE if $F(x,y)$
uld be a homogenous fn of degree '0'.

e : $Mdx + Ndy = 0$ is said to be homogenous
if all the terms of M & N should be
same degree.

for variable - seperable sub $y = vx$ (or) $x = vy$.

$$x\frac{dy}{dx} = y\left\{\log y - \log x + 1\right\}$$

$$\frac{dy}{dx} = \frac{y}{x}\left\{\log\left(\frac{y}{x}\right) + 1\right\}$$

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

$$v + x\frac{dv}{dx} = v\left\{\log v + 1\right\}.$$

$$x\frac{dv}{dx} = v\log v.$$

$$\int\frac{1}{v\log v}\,dv = \int\frac{1}{x}\,dx$$

$$\log(\log v) = \log x + \log c.$$

$$\log v = xc. \Rightarrow v = e^{cx} \Rightarrow y = xe^{cx}.$$

$$(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0.$$

Non - homogenous DE :-

An eq^n of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

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case (i) :- $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

In this case there exists a substitution which reduces the given eqn to variable - seperable form.

genous

be

y.

case (ii) :- $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

sub, $x = X + h$, $y = Y + k$

$dx = dX$, $dy = dY$

$$\frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y + (a_1 h + b_1 k + c_1)}{a_2 X + b_2 Y + (a_2 h + b_2 k + c_2)}$$

choose h, k so that $a_1 h + b_1 k + c = 0$, $a_2 h + b_2 k + c_2 =$

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y}{a_2 X + b_2 Y}.$$

⊛ $(2x + 2y - 1)\, dx = (x + y + 1)\, dy$

$$\frac{dy}{dx} = \frac{2x + 2y - 1}{x + y + 1} \qquad \left(\frac{a_1}{a_2} = \frac{b_1}{b_2}\right)$$

$x + y = v$

$1 + \frac{dy}{dx} = \frac{dv}{dx}$

$$\frac{dv}{dx} - 1 = \frac{2v - 1}{v + 1}$$

$$\frac{dv}{dx} = \frac{3v}{v + 1}$$

$$\int \frac{v+1}{v}\, dv = 3\int dx \Rightarrow v + \log v = 3x + c.$$

$$x + y + \log(x+y) = 3x + c \Rightarrow y - 2x + \log(x+y) = c$$

⊛ which of the following sub reduces the non - homogenous eqn $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$ to hom. form.

Sol $a_1 = 1$ $b_1 = 1$ $a_1 h + b_1 k + c_1 = 0$

$k + h - 2 = 0 \qquad k = 3 \qquad h = -1$

$k - h - 4 = 0$

$x = (X + 1) \qquad y = (Y + 3)$

hom form $\Rightarrow \dfrac{dY}{dX} = \dfrac{Y + X}{Y - X}$.

$\Rightarrow$ Exact differential eq^n :-

An eq^n $Mdx + Ndy = 0$ is said to be an exact DE if

$\exists\, f(x,y)$ such that $d[f(x,y)] = Mdx + N\,dy$.

condition to be exact $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

if this satisfies then find soln.

$$Mdx + Ndy = 0$$

$$\underset{\text{Keep 'y' const}}{\int M\,dx} + \int \begin{pmatrix}\text{terms of } N \\ \text{without } x\end{pmatrix} dy = c$$

⊛ $\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\} dx + \left\{x + \log x - x \sin y\right\} dy = 0$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

exact//

soln.

$$\int_{y \to \text{const}} \left(y(1 + 1/x) + \cos y\right) dx + \int 0 \cdot dy = c$$

$$= y(x + \log x) + x\cos y + 0 = c.$$

⊛ $y\sin 2x\,dx - (1+y^2+\cos^2 x)\,dy = 0$

M N

$$\frac{\partial M}{\partial y} = \sin 2x \qquad \frac{\partial N}{\partial x} = 2\sin x\cos x$$

$$= \sin 2x$$

eqn is exact

be

$$\int y\left(\frac{-\cos 2x}{2}\right) + \int (-1-y^2)\,dy = c.$$

$$-y\frac{\cos 2x}{2} - y - \frac{y^3}{3} = c$$

⊛ The eq $py\,dx + (1+\sin^2 y + \cos^2 x)\,dy = 0$ is exact then

a) $p = \cos 2x$ b) $p = \sin 2x$

c) $p = -\cos 2x$ d) ✓ $p = -\sin 2x$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$p = -\sin 2x$$

⊛ $p\,dx + (1+\cos^2 x + \sin^2 y)\,dy = 0$ is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(p) = 2\cos x(-\sin x)$$

$$= -\sin 2x$$

$$p = \int -\sin 2x\,dy$$

$$p = -y\sin 2x$$

⊛ The DE

$$(3a^2x^2 + by\cos x)\,dx + (2\sin x - 4ay^3)\,dy = 0$$

is exact then

a) exactness depends on both a & b

b) " not

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$+b\,\cancel{\sin x}\;\cos x = 2\cos x$$

$b = +2$ not depends on 'a' only depends on b.

⊛ $y\,dx - x\,dy = 0$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1$$

not exact.

to make it exact ; multiply with $1/y^2$

$$\frac{1}{y}\,dx - \frac{x}{y^2}\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2} = \frac{\partial N}{\partial x} \quad \text{(exact)}$$

$\therefore$ $1/y^2$ is called integrating factor.

⇒ Integrating factor :- A non-exact eq^n is converted to exact by multiplying a fn $f(x,y)$ then $f(x,y)$ is called integrating factor.

The integrating factors of $y\,dx - x\,dy = 0$ are

$$\frac{1}{y^2},\ \frac{1}{x^2},\ \frac{1}{xy},\ \frac{1}{x^2+y^2}\ \ldots$$

$$\boxed{M\,dx + N\,dy = 0}$$

non-exact eq $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

I — All the terms of M & N should be of same degree, then

II — eq^n is of the form ⊛ $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$

IV — $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ only

$\int f(x)\,dx$

IV $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$ only (or) constant

$$IF = e^{\int g(y)\,dy}$$

V Inspection method.

(*) $y(xy-2)\,dx + x(x^2y^2 + 2xy + 1)\,dy = 0$. —①

$(xy^2 - 2y)\,dx + (x^3y^2 + 2x^2y + x)\,dy = 0$

$\dfrac{\partial M}{\partial y} = 2xy - 2$

$\dfrac{\partial N}{\partial x} = 3x^2y^2 + 4xy + 1$

from ① it is in form $y f(xy)\,dx + x\,g(xy)\,dy = 0$

$\boxed{\begin{array}{l} f(xy) = x^2y^2,\ xy \ldots \\ f(x,y) = x^2y^2 + 2x \ldots \end{array}}$

it is form II. so $IF = \dfrac{1}{Mx - Ny}$

$$IF = \frac{1}{x^2y^2 - 2xy - (x^3y^3 + 2x^2y^2 + xy)}$$

$$= \frac{-1}{x^3y^3 + 3x^2y^2 + 3xy}$$

(*) The non-exact homogenous DE $M\,dx + N\,dy = 0$ is converted to exact by multiplying with which of following fns.

Ans: $\dfrac{1}{Mx + Ny}$.

given homogenous so degree of M & N will be same so use II.

(*) $x^2y\,dx - (x^3 + y^3)\,dy = 0$.

$f(x)$ only

$\dfrac{\partial M}{\partial y} = x^2$ $\qquad \dfrac{\partial N}{\partial x} = -3x^2$

all the M & N have same degree

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$x)\,dx$

$$\frac{x^2y}{-y^4}\,dx - \frac{(x^3+y^3)}{-y^4}\,dy = 0 \quad \rightarrow \text{exact}$$

$\int_{y\text{-const}} M\,dx + \int \text{terms no } x$

$$-\frac{x^3}{3y^3} + \log y = c \quad \text{is the soln.}$$

$$x(x-2y)\,dy + (x^2+y^2+1)\,dx = 0$$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2x - 2y$$

$$\cancel{\frac{\partial M}{\partial y}} \; \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - 2x + 2y}{x(x-2y)}$$

$$= \frac{-2(x-2y)}{x(x-2y)}$$

$$= \frac{-2}{x} \quad f(x)$$

$$IF = e^{-\int \frac{2}{x}} = e^{-2\log x} = \frac{1}{x^2}$$

$$\frac{x(x-2y)}{x^2}\,dy + \frac{(x^2+y^2+1)}{x^2}\,dx = 0$$

soln :- $x - \frac{y^2}{x} - \frac{1}{x} + \int 1\,dy$

$$x - \frac{(1+y^2)}{x} + y = c$$

$f(x)$	$e^{\int f(x)\,dx}$
$\frac{1}{x}$	x
$\frac{2}{x}$	x^2
$\frac{3}{x}$	x^3
$\frac{-1}{x}$	$x^{-1} = 1/x$
$\frac{-2}{x}$	$1/x^2$
$-$	$1/_{3}$

⊛ $y(1-xy)\,dx - x(1+xy)\,dy = 0$

$$\frac{\partial M}{\partial y} = 1-2xy \qquad \frac{\partial N}{\partial x} = -1-2xy.$$

case (ii)

$$Mx - Ny = xy - x^2y^2 + xy + x^2y^2$$

$$= 2xy.$$

$$IF = \frac{1}{2xy}.$$

$$\frac{y(1-xy)}{\cancel{2}xy} - \frac{x(1+xy)}{\cancel{2}xy} = 0 \qquad \left(\begin{array}{l}\text{don't cancel}\\ \text{variables -- cancel}\\ \text{constents}\end{array}\right)$$

$$\frac{1-xy}{x} - \frac{1+xy}{y} = 0$$

soln is $\log x - xy - \int \frac{1}{y}\,dy$

$$\log x - xy - \log y = c.$$

⊛ By multiplying with which of the following functions the eq $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$ is converted to exact

$$\frac{\partial M}{\partial y} = 4y^3+2 \qquad \frac{\partial N}{\partial x} = y^3-4$$

case ④

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y^3-4-4y^3-2}{y^4+2y}$$

$$= \frac{-3(y^3+2)}{y(y^3+2)}$$

$$= \frac{-3}{y}.$$

$$IF = 1/y^3 \quad \text{(or)} \quad e^{\int -3/y} = e^{-3\log y} = \frac{1}{y^3}$$

⊛ The IF of

$$y(x^2y^2+xy+1)\,dx + x(x^2y^2-xy+1)\,dy = 0$$

case ② since $x f(xy) + y\, g(xy) = 0$.

$$Mx - Ny = x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy$$

$$= 2x^2y^2 .$$

$$IF = \frac{1}{2x^2y^2} \quad (\text{or}) \quad \frac{1}{x^2y^2}$$ (const doesnot make differen)

(*) $y\,dx - x\,dy + (1+x^2)\,dx + x^2\sin y\,dy = 0$.

$\cancel{(x^2+y+1)\,dx} + \cancel{(-x + x^2\sin y)\,dy} = \cancel{0}$.

when there is $y\,dx - x\,dy$ is a part of DE then simply multiply with $1/x^2$, $1/y^2$, $\frac{1}{xy}$, $\frac{1}{x^2+y^2}$ & then integrate.

By multiplying with $1/x^2$ we can easily integrate it is not possible with other terms

Depending on the DE multiply with IF.

Note:

$$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$$

$$\frac{y\,dx - x\,dy}{x^2} = -d\left(\frac{x}{y}\right)$$

$$\frac{y\,dx - x\,dy}{xy} = d\left[\log\left(\frac{x}{y}\right)\right].$$

$$\frac{y\,dx - x\,dy}{x^2+y^2} = d\left[\tan^{-1}\left(\frac{x}{y}\right)\right].$$

$\int y e^{y} = e^{y}(y-1)$

$\int y e^{-y} = -e^{-y}(y+1)$

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sol multiply with $1/x^2$

$$\frac{y\,dx - x\,dy}{x^2} + \left(\frac{1+x^2}{x^2}\right)dx + \frac{\cancel{x^2}\sin y}{\cancel{x^2}}dy = 0$$

$= \frac{-x}{y} - \frac{1}{x} + x + (-\cos y) = c$

$-xy$

⊛ $y\,dx - x\,dy + y(1+x^2)\,dx + xy^2 e^y\,dy = 0$

multiply with $1/xy$

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$\frac{y\,dx - x\,dy}{xy} + \left(\frac{1+x^2}{x}\right)dx + ye^y\,dy = 0$

$\int d\left[\log\left(\frac{x}{y}\right)\right] + \int\left(\frac{1}{x} + x\right)dx + \int ye^y\,dy = 0$

$\log\left(\frac{x}{y}\right) + \log x + \frac{x^2}{2} + e^y(y-1) = c$

⊛ $\boxed{y\,dx + x\,dy = d(xy)}$

$\boxed{\frac{y\,dx + x\,dy}{xy} = d\left[\log(xy)\right]}$

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⊛ $y\,dx + x\,dy + xy^2 e^{-y}\,dy = 0$

multiply with $1/xy$

$\int \frac{y\,dx + x\,dy}{xy} + \int ye^{-y}\,dy = 0$

$\log(xy) - e^{-y}(y+1) = c$

$e^y(y-1)$

$e^{-y}(y+1)$

⇒ Linear Dequations :-

A DE is said to be linear if the dependent variable & derivatives should be of 1^{st} degree only & there should be no product of them.

(should) $(y)^1, \left(\frac{dy}{dx}\right)^1, \left(\frac{d^2y}{dx^2}\right)^1 \ldots\ldots$ degree = 1.

(should not) $y\frac{dy}{dx}, \frac{d^2y}{dx^2}\frac{d^4y}{dx^4} \ldots$ (there should not be multiplication)

$$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right) + e^y = 0$$

is not a linear eq^n becoz $\left(e^y = 1 + y + \frac{y^2}{} + y^3 \ldots\right)$ multiplication came

$$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 + 4y = 0$$ is non-linear eq.

$\frac{dy}{dx}\cdot\frac{dy}{dx}$ multiplication came.

⇒ Linear in y :-

$$\boxed{\frac{dy}{dx} + Py = Q}$$

P & Q are fns of x or constants.

↓ soln.

IF

$$\boxed{y\, e^{\int P dx} = \int Q\, e^{\int P dx}\, dx + C}$$

⇒ Linear in x :-

$$\boxed{\frac{dx}{dy} + Px = Q}$$

P & Q are fns of y or constants

↓ soln is.

IF

$$\boxed{x\, e^{\int P dy} = \int Q\, e^{\int P dy} + C}$$

) $x\cos x \frac{dy}{dx} + y(x\sin x + \cos x) = 1.$

$$\frac{dy}{dx} + y\tan x + \frac{y}{x} = \frac{1}{x\cos x}$$

$$\frac{dy}{dx} + \underbrace{y\left(\tan x + \frac{1}{x}\right)}_{P} = \underbrace{\frac{1}{x\cos x}}_{Q}$$

$$IF = e^{\int\left(\tan x + \frac{1}{x}\right)dx} = e^{\log\sec x + \log x}$$

soln

soln

soln

lication came

(y^2+y^3)

ar eq.

of x

fns

stants

soln :- $y(x\sec x) = \int \frac{1}{\cancel{x}\cos x} \cdot \cancel{x}\sec x \, dx + c$

$= \int \sec^2 x \, dx$

$xy\sec x = \tan x + c$

⊛ $(x + 2y^3)\frac{dy}{dx} = y$ with $x(1) = 0$

linear in x.

$x + 2y^3 = y\frac{dx}{dy}$

$\frac{dx}{dy} = \frac{x + 2y^3}{y}$

$\frac{dx}{dy} - \frac{x}{y} = 2y^2 \qquad P = \frac{-1}{y}$

$IF = e^{-\int 1/y} = 1/y$

soln: $x \cdot \frac{1}{y} = \int 2y^{\cancel{2}} \cdot \frac{1}{\cancel{y}} + c$

$\frac{x}{y} = y^2 + c \qquad$ at $y = 1,\ x = 0$

$\frac{0}{1} = 1 + c \qquad c = -1$

$x/y = y^2 - 1 \cdot y$

⊛ $x^2\frac{dy}{dx} + 2xy = \frac{2\log x}{x}$; $y(1) = 0$ find y when $x = e$.

linear in y.

$\frac{dy}{dx} + \frac{2y}{x} = \frac{2\log x}{x^3}$

$IF = e^{\int \frac{2}{x}} = x^2$

soln $\quad yx^2 = \int \frac{2\log x}{x^{\cancel{3}}} \cdot \cancel{x^2}$

$\int f(x)^n \cdot f'(x) = f(x)^{n+1}$

at $x = 1$; $y = 0$ $\Rightarrow c = 0$

$$x^2 y = (\log x)^2 + 0$$

at $x = e$ $\quad e^2 y = (\log e)^2$

$$e^2 y = 1 \Rightarrow y = 1/e^2$$

(8) $x \frac{dy}{dx} + y = x^4$; $y(1) = \frac{6}{5}$

$$\frac{dy}{dx} + \frac{y}{x} = x^3$$

$$IF = e^{\int 1/x} = \cancel{\frac{d}{x}} \cdot x$$

<u>soln</u>

$$\frac{-y}{\cancel{x}} x = \int x^3 \cdot \frac{-x}{\cancel{x}} + c$$

$$\frac{-y}{\cancel{x}} x = -\frac{x^5}{5} + c$$

at $x = 1$ $y = 6/5$ $\Rightarrow c = 1$

$$xy = \frac{x^5}{5} + 1$$

$$y = \frac{x^4}{5} + \frac{1}{x}$$

$\Rightarrow$ $\frac{dy}{dx} + Py = Qy^n$ $\rightarrow$ is not a linear eqn

sub $(y^{1-n} = v)$ reduces Bernouli eq to linear eqn

soln.

$$y^{1-n} e^{\int (1-n) P\,dx} = \int (1-n) Q\, e^{\int (1-n)\,dx} dx + c.$$

(If y^n is there is q^n ~~see~~ multiply $(1-n)$ to every term in soln)

(21) The sub $y^{1-n} = v$ reduces the non-linear eq

$\frac{dy}{dx} + Py = Qy^n$ to which of following linear

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{Py}{y^n} = Q.$$

$$y^{1-n} = v \Rightarrow (1-n)y^{1-n-1}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}.$$

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q.$$

$$\boxed{\frac{dv}{dx} + Pv(1-n) = Q(1-n).}$$

* which of the following sub reduces the non-linear eq^n to linear form

$$y\frac{dy}{dx} + x^2y^3 = x^3y^3.$$

$$\frac{dy}{dx} + x^2y^2 = x^3y^2$$

$$y^{1-n} = v \Rightarrow y^{1-2} = v \Rightarrow y^{-1} = v.$$

* $$\frac{dy}{dx} - y\tan x = -y^2\sec x.$$

ear eq^n

$$e^{\int(1-2)(-\tan x)} = e^{\log\sec x} = \sec x.$$

$$y^{1-2}\sec x = \int(1-2)(-\sec x)\sec x\,dx$$

n soln)

$$\frac{\sec x}{y} = \tan x + c.$$

eq

$\Rightarrow$ Equation of the form

inear

$$\boxed{f'(y)\frac{dy}{dx} + Pf(y) = Q}$$

$$f(y) = v \Rightarrow f'(y)\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + Pv = Q$$

⊛ The sub $(\log z)^{-1} = v$ reduces the non-linear eq

$$\frac{dz}{dx} + \frac{z\log z}{x} = \frac{z(\log z)^2}{x^2}$$

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{(\log z)^{-1}}{x} = \frac{1}{x^2}$$

$$(\log z)^{-1} = v \Rightarrow -\frac{(\log z)^{-2}}{z}\frac{dz}{dx} = \frac{dv}{dx}$$

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} = -\frac{dv}{dx}$$

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}$$

$$\frac{dv}{dx} - \frac{v}{x^2} = \frac{-1}{x^2}$$

$$x\frac{dy}{dx} + y\log y = xye^x$$

$$\frac{1}{y}\frac{dy}{dx} + \frac{\log y}{x} = e^x$$

$$\log y = v \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + \frac{v}{x} = e^x$$

$$IF = e^{\int \frac{1}{x}} = x.$$

$$v\,x = \int e^x . x \; dx$$

$$x \log y = e^x(x-1) + c.$$

⇒ Clairaut's eqⁿ :-

An eq of the form

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right).$$

-linear

or

$$y = px + f(p)$$

replace $\frac{dy}{dx} \rightarrow c.$

$\boxed{y = cx + f(c)}$ is the fn of soln.

⊛ $p = \sin(y - xp)$

$$y - xp = \sin^{-1} p$$

$$y = xp + \sin^{-1} p$$

$$y = cx + \sin^{-1} c \quad (c \text{ is a const}).$$

⊛ $(x-a)y'^2 + (x-y)y' - y = 0$

$$xy'^2 - ay'^2 + xy' - yy' - y = 0$$

$$xy'(y'+1) - ay'^2 = y(y'+1)$$

$$xy' - \frac{ay'^2}{(y'+1)} = y$$

$$y = cx - \frac{ac^2}{1+c^2}$$

⊛ $\left(y - x\frac{dy}{dx}\right)\left(\frac{dy}{dx} - 1\right) = \frac{dy}{dx}.$

$$(y - xy')(y' - 1) = y'$$

⇒ Higher order linear eq^n with constant coefficients

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

$k_1, k_2 \ldots k_{n-1}$ are constants

X is fn of x.

$\boxed{D = \frac{d}{dx}}$ — differential operator.

$\boxed{\frac{1}{D} = \int}$ — inverse differential operator.

$$\left(D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n\right) y = X$$

$$F(D)\, y = X.$$

$\boxed{F(D) = D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n = 0}$

is called auxillary eq^n.

- The complete solution of $F(D)\,y = X$ is

y = complementary fn (CF) + particular integral (PI)

- If $X = 0$ then $F(D)y = 0$ is the called the homogenous linear DE.

- If $X \neq 0$ then $F(D)y = X$ is called non-homogenous near DE

- The soln of homogenous linear DE i.e. $(F(D)\,y = 0)$ is called CF.

The no. of arbitrary constants in the CF should be equal to the higher order of given DE.

PI will not contain any arbitrary constants.

(A) If $X = 0$ the complete soln of $F(D)y = X$ is only CF.

⇒ **Procedure to find the CF :-**

By assuming 'D' as an algebraic quantity. $F(D) = 0$ becomes an algebraic eq^n & by solving it we get roots for these. Based on the nature of these roots we write CF as follows.

Nature of roots	CF
1. Real & Distinct $D = m_1, m_2, m_3$	$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2. Real & repeated $D = m_1, m_1, m_3$	$CF = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x}$
3. complex & distinct $D = a \pm ib, m_3$	$e^{ax}[c_1 \cos bx + c_2 \sin bx] + c_3 e^{m_3 x}$
4. complex & repeated $D = a \pm ib, a \pm ib, m_5$	$e^{ax}[(c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx] + c_5 e^{m_5 x}$

Eg:

roots

$D = 1, -1/2, 1/2$ $\quad CF = c_1 e^{x} + c_2 e^{-1/2 x} + c_3 e^{x/2}$

$D = -2, -2, 1$ $\quad CF = (c_1 + c_2 x) e^{-2x} + c_3 e^{x}$

$D = 3 \pm 4i, -1$ $\quad CF = e^{3x}[c_1 \cos 4x + c_2 \sin 4x] + c_3 e^{-x}$

$D = \pm 2i, \pm 3$ $\quad CF = [c_1 \cos 2x + c_2 \sin 2x]$

G

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$D^2 - 5D + 6 = 0 \qquad D = 3, 2$$

$$CF = c_1 e^{3x} + c_2 e^{2x}.$$

$$\frac{d^4y}{dx^4} - 81y = 0$$

$$D^4 - 81 = 0$$

$$(D^2 - 9)(D^2 + 9) = 0$$

$$D = \pm 3, \pm i3$$

$$y = c_1 e^{3x} + c_2 e^{-3x} + (c_3 \cos 3x + c_4 \sin 3x)$$

in option it ~~will~~ maybe $e^{-3x} + \sin 3x$ choosing different values of c_1, c_2, c_3, c_4 ...

$$y'' + 2py' + (p^2+q^2)y = 0$$

$$D^2 + 2pD + (p^2+q^2) = 0$$

$$D = \frac{-2p \pm \sqrt{4p^2 - 4(p^2+q^2)}}{2}$$

$$= -p \pm \sqrt{p^2 - p^2 - q^2}$$

$$= -p \pm iq$$

$$y = e^{-xp}(c_1 \cos qx + c_2 \sin qx).$$

$$y''' - 4y'' + 5y' - 2y = 0$$

$$D^3 - 4D^2 + 5D - 2 = 0$$

$D = 1$ is root.

1	1	-4	5	-2
	0	1	-3	2

$D^2 - 3D + 2 = 0$

$D = +1, +2$

$+2$ / $-2 \quad -1$

$D^2 - 2D - D + 2 = 0$

$D(D-2) - 1(D-2) = 0$

$(D-1)(D-2) = 0$

roots are $1, +1, +2$

$y = c_1 e^{x} + c_2 e^{x} + c_3 e^{2x}$

$y = (c_1 + c_2 x)e^{x} + c_3 e^{2x}$

:- maybe

values

Note :- If $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots$ is a soln of $F(D)y = 0$ then each one $y_1, y_2, y_3 \cdots$ is linearly independent soln of $F(D)y = 0$.

3x)

⊛ $\frac{d^2 f}{dt^2} - 4\frac{df}{dt} + 4f = 0$

a) $f_1 = e^{2t}, f_2 = e^{-2t}$ ~~b)~~ $f_1 = e^{2t} ; f_2 = te^{2t}$

c) $f_1 = e^{-2t} ; f_2 = te^{-2t}$ d) $f_1 = e^{-2t} ; f_2 = te^{2t}$

$D^2 - 4D + 4 = 0$

4 / $-2 \quad -2$

$(D-2)^2 = 0$

$D = 2, 2$

$y = (c_1 + c_2 t)e^{2t}$

$c_1 e^{2t} + c_2 t e^{2t}$

⊛ The DE $(D^2 + 4\cos x\, D + 3)y = x^2$ is b

a) homogenous DE b) non-homogenous DE with const coef

~~c)~~ Linear DE d) Non-Linear DE

⊛ y_1, y_2 are linearly independent solns of $F(D)y = 0$ then which of the following is the soln of the same eq^n.

⊛ y_1, y_2 are linearly independent soln of the corresponding homogenous eqn of $F(D)y = x$ then $y_1 + y_2$ is the soln of which of following eqn

a) $F(D)y = x$ b) $F(D)y = 0$

c) $x = 0$ ~~d)~~ $F(D)y = 0$

⊛ e^{2x}, e^{-3x} are the ~~so~~ linearly independent solns

$$D = 2, -3$$

$$(D-2)(D+3)y = 0$$

$$(D^2+D-6)y = 0$$

$$y'' + y' - 6y = 0$$

⊛ $e^{2x}, x e^{2x}$

$$= (c_1 + c_2 x)e^{2x}$$

$$D = 2, 2$$

$$(D-2)^2 y = 0$$

$$(D^2 - 4D + 4)y = 0 \Rightarrow y'' - 4y' + 4y = 0$$

⊛ e^{2x}, e^{-2x} are soln of which ...

(see options sub $D = 2$ & -2 which satisfies the eq) it may be

$$D = 2, -2$$

$$(D^2-4)y = 0 \Rightarrow y'' - 4y = 0 \quad \frac{d^4y}{dx^4} - 16y = 0$$

⊛ $\sin 3x, \cos 3x$ are LI soln of which of the following DE

$$y = c_1 \cos 3x + c_2 \sin 3x$$

the

then

eqn

solns

such

$S_cy = 0$

the

$$y'' + 4y' + 13y = 0 \; ; \quad y(0) = 0 \; , \quad y'(0) = 1.$$

$$(D^2 + 4D + 13) = 0.$$

$$D = \frac{-4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{-4 \pm i6}{2} = -2 \pm 3i.$$

$$y = e^{-2x}\left(c_1 \cos 3x + c_2 \sin 3x\right).$$

at $x = 0$; $y = 0$.

$$0 = c_1$$

$$y = e^{-2x} c_2 \sin 3x.$$

$$y' = c_2\left[e^{-2x}(3\cos 3x) + (-2)e^{-2x}\sin 3x\right]$$

at $x = 0$ $y' = 1$

$$1 = c_2[3] \Rightarrow c_2 = 1/3.$$

$$y = \frac{e^{-2x}}{3} \sin 3x.$$

which of the following is not a soln of

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0.$$

where $R^2 C = 4L$ & R, L, C be the constants

a) $i = e^{-Rt/2L}$ b) $i = te^{-Rt/2L}$

~~c)~~ $i = te^{Rt/2L}$ d) $i = e^{-Rt/2L} + te^{-Rt/2L}$

$$D^2 + \frac{R}{L} D + \frac{1}{LC} = 0$$

$$D = \frac{-R/L \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$D = \frac{-R}{2L}, \frac{-R}{2L}.$$

$$y = (c_1 + c_2 t) e^{-R/2L}$$

$$= e^{-R/2L} + t e^{-R/2L}$$

⊛ $\frac{d^2y}{dx^2} + \omega^2 y = 0$; $y(0) = 0$, $y(l) = 0$

a) $y = \sum_n c_n \cos \frac{n\pi x}{l}$ b) $y = \sum_n c_n e^{\frac{n\pi x}{l}}$

~~c)~~ $y = \sum_n c_n \sin \frac{n\pi x}{l}$ d) $y = \sum_n c_n x^{\frac{n\pi}{l}}$

Sol sub $y = 0$ at $x = 0$ & $x = l \Rightarrow y = 0$.

⊛ $9y'' - 6y' + y = 0$; $y(0) = 3$, $y'(0) = 1$

$(9D^2 - 6D + 1) y = 0$

$$D = \frac{+6 \pm \sqrt{36 - 36}}{18}$$

$$D = \frac{1}{3}, 1/3.$$

$$y = (c_1 + c_2 x) e^{x/3}$$

$y(0) = 3 \Rightarrow 3 = c_1$.

$y'(0) = 1 \Rightarrow y' = c_2 e^{x/3} + (c_1 + c_2 x) e^{x/3} \cdot \frac{1}{3}$

$\Rightarrow 1 = c_2 + 3c_1$

$\Rightarrow c_2 = 0$

$$y = (3) e^{x/3}$$

⊛ For what value of 'λ' the DE $(D^2 + \lambda) y = 0$ will have non-trivial soln.

$y(0) = 0$; $y(\pi) = 0$

$$y = (c_1 \cos\sqrt{\lambda} x + c_2 \sin\sqrt{\lambda} x)$$

$$y(0) = 0 \Rightarrow 0 = c_1$$

$$y(\pi) = 0 \Rightarrow 0 = c_2 \sin \pi\sqrt{\lambda}$$

$$\sin \pi\sqrt{\lambda} = 0$$

$$\pi\lambda = n^2\pi$$

$$\lambda = 1, 4, 9, 16 \ldots$$

Q (a) The complete soln of DE $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$ is $y = c_1 e^{-x} + c_2 e^{-3x}$. p, q are:

$$D = -1, -3$$

$$(D+1)(D+3)\, y = 0$$

$$(D^2 + 4D + 3)\, y = 0$$

$$\frac{d^2y}{dx^2} + \frac{4dy}{dy} + 3y = 0 \qquad p = 4,\ q = 3.$$

(b) The soln of $y'' + py' + (q+1)y = 0$ is

$$y'' + 4y' + 4y = 0$$

$$(D+2)^2 y = 0$$

$$D = -2, -2 \Rightarrow y = (c_1 + c_2 x) e^{-2x}.$$

$$\frac{d^2 n}{dx^2} + \frac{-n}{L^2} = 0; \quad n(0) = K, \quad n(\infty) = 0.$$

$$\left(D^2 - \frac{1}{L^2}\right) n = 0.$$

$$D = \pm\frac{1}{L} \qquad \cancel{y = Ke^{x/L}}$$

$$y = c_1 e^{x/L} + c_2 e^{-x/L}.$$

sub. conditions:

Particular integral :-

$$F(D)\,y = X.$$

$$PI = \left[\frac{1}{F(D)}\right] X.$$

X may be

- I: e^{ax} @ constant
- II: $\sin ax$ @ $\cos ax$
- III: x^m
- IV: $e^{ax}.V$
  - V can be
    - $\sin bx$ @ $\cos bx$
    - x^m

case (I) : $X = e^{ax}$ @ constant

$$PI = \left[\frac{1}{F(D)}\right] e^{ax}.$$

replace 'D' by 'a' in $F(D)$

$$\boxed{PI = \left[\frac{1}{F(a)}\right] e^{ax}} \qquad (F(a) \neq 0)$$

if $F(a) = 0$:-

$$PI = x\left[\frac{1}{F'(D)}\right] e^{ax}$$

replace $D \to a$ in $F'(D)$

$$\boxed{PI = x\left[\frac{1}{F'(a)}\right] e^{ax}}$$

---

if $F'(a) = 0$

$$PI = x^2\left[\frac{1}{F''(a)}\right] e^{ax}$$

* Find PI of $y'' + 3y' - 2y = e^{2x} + 3$

$$PI = \left[\frac{1}{D^2+3D-2}\right]\left[e^{2x}+3\right] \qquad 3 = 3e^{0x}$$

$$= \frac{1}{2^2+3(2)-2}e^{2x} + \frac{3}{0^2+3(0)-2}e^{0x}$$

$$= \frac{e^{2x}}{8} - \frac{3}{2}.$$

he

* PI of $y'' + 5y' + 6y = e^{-3x}$

$$PI = \left[\frac{1}{D^2+5D+6}\right]e^{-3x} \qquad (\text{Dr is } 0).$$

so differentiate

$$= x\left(\frac{1}{2D+5}\right)e^{-3x}$$

$$= -xe^{-3x}$$

* PI of $y'' + 4y' + 4y = e^{2x}$.

$$PI = \left[\frac{1}{D^2+4D+4}\right]e^{-2x} \qquad (\text{Dr is } 0)$$

$$= \left(\frac{x}{2D+4}\right)e^{-2x} \qquad (\text{Dr is } 0)$$

$$= \frac{x^2}{2}e^{-2x}$$

case (2) :- $X = \sin ax$ (or) $\cos ax$ (or) $\sin(ax+b)$ (or) $\cos(ax+b)$

$$PI = \left[\frac{1}{F(D)}\right]\sin ax$$

replace D^2 by $-a^2$ in $F(D)$.

$$= \frac{1}{}\sin ax.$$

if $F(-a^2) = 0$

then $PI = x\left[\frac{1}{F'(D)}\right] \sin ax$

$$PI = x \frac{1}{F'(-a^2)} \sin ax$$

⊛ $\frac{d^2y}{dx^2} + 5y = \cos(3x+2)$

$$PI = \left[\frac{1}{D^2+5}\right] \cos(3x+2)$$

$$= \frac{1}{-9+5} \cos(3x+2)$$

$$= \frac{-\cos(3x+2)}{4}$$

⊛ $y'' + 4y = \sin 2x$

$$PI = \left[\frac{1}{D^2+4}\right] \sin 2x \qquad (Dr = 0)$$

$$= \frac{x}{2D} \sin 2x = \frac{x}{2}\left[\frac{1}{D} \sin 2x\right]$$

Don't do integ on x only do on $\sin 2x$.

$$= \frac{x}{2}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{-x\cos 2x}{4}$$

⊛ $(D^3 + D^2 + 2D + 2)y = \cos x$

$$PI = \left[\frac{1}{D^3 + D^2 + 2D + 2}\right] \cos x$$

replace $D^2 = -a^2 = -1$

$$= \frac{1}{-D - 1 + 2D + 2} \cos x \qquad (D^3 = D \cdot D^2)$$

$$= \frac{D-1}{(D^2} \quad \cos x)$$

$$= \frac{D-1}{-1-1} \cos x = \frac{(D-1)}{-2} \cos x$$

$$= \frac{1}{2}\left(\cos x + D(\cos x)\right)$$

$$= \frac{1}{2}\left\{\cos x - \sin x\right\}$$

case : (3) :- $X = x^m$ or poly in x (m is +ve).

$$PI = \left[\frac{1}{F(D)}\right] x^m$$

$$= \frac{1}{*\left[1+\phi(D)\right]} x^m$$

($*$ is the least power in $F(D)$)

$$= \frac{1}{*}\left[1+\phi(D)\right]^{-1} x^m$$

Ex $\frac{d^2y}{dx^2} + 2y = x^3$

$$PI = \left(\frac{1}{D^2+2}\right) x^3$$

$$= \frac{1}{2}\left(\frac{1}{1+0.5D^2}\right) x^3$$

$$= \frac{1}{2}\left[1+\frac{D^2}{2}\right]^{-1} x^3$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\frac{D^4}{4}\cdots\right] x^3$$

$$= \frac{1}{2}\left[x^3 - 3x + 0 \cdots\right] = \frac{x^3-3x}{2}$$

Ex $D^2 + D = x^2+2$

$$PI = \left(\frac{1}{D^2+D}\right)(x^2+2) \quad \cancel{\frac{1}{D^2+D}x^2} + \cancel{\left(\frac{1}{D^2+D}\right) 2e^{0x}}$$

]

on x

$\sin 2x$

2).

$$= \frac{1}{D}\left(1+\frac{D}{D}\right. \quad \left.+2\right)$$

$$= \frac{1}{D}(1+D)^{-1}(x^2+\ )$$

$$= \frac{1}{D}(1-D+D^2)(x^2+2)$$

$$= \frac{1}{D}\{x^2+2-2x+2)$$

$$= \frac{x^3}{3}+4x-x^2$$

Case (4) :- $X = e^{ax}.V$

$(v = \cos bx$ (or) $\sin bx)$ (or) $(v = x^m$ (or) poly in $x)$

$$PI = \left[\frac{1}{F(D)}\right] e^{ax} V.$$

replace $D \to$ '$D+a$' in $F(D)$.

$$= e^{ax}\left[\frac{1}{F(D+a)}\right] V.$$

$\left[\frac{1}{F(D+a)}\right] V$ → case (2) if $V = \sin bx$ (or) $\cos bx$

→ case (3). if $v = x^m$.

(*) $$\frac{d^2y}{dx^2} + 3y = e^x \sin 2x.$$

$$PI = \left[\frac{1}{D^2+3}\right] e^x \sin 2x$$

$$= e^x\left[\frac{1}{(D+1)^2+3}\right] \sin 2x$$

$$= e^x\left[\frac{1}{D^2+2D+4}\right] \sin 2x$$

$$= \frac{e^x}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{e^x \cos 2x}{4}$$

poly in x

(*) $\frac{d^2y}{dx^2} - 4y = \cos^2 x$

$F(D) = D^2 - 4 = 0 \quad D = \pm 2$

$CF = c_1 e^{2x} + c_2 e^{-2x}$

$$PI = \left(\frac{1}{D^2-4}\right)\cos^2 x = \frac{1}{2}\left[\frac{1}{D^2-4}\right](\overset{e^{0x}}{1} + \cos 2x)$$

$$= \frac{1}{2}\left[\frac{1}{-4}\right] + \frac{1}{2}\left[\frac{1}{-4-4}\right]\cos 2x$$

$$PI = \frac{-1}{8} - \frac{1}{16}\cos 2x$$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} - \frac{1}{16}\cos 2x.$$

$\cos bx$

(*) The PI of $\frac{d^2y}{dx^2} + a^2y = \sin ax$ is

$$PI = \left(\frac{1}{D^2+a^2}\right)\sin ax = \frac{x}{2D}\sin ax$$

$$= \frac{-x}{2a}\cos ax.$$

(*) The PI of $\frac{d^2y}{dx^2} + 4y = \sin 2x + \cos 3x$ is

$(Ax\cos 2x + B \underset{\cos}{\cancel{\sin}} 3x) \quad A, B = \_\_$

$(PI)_1 = \frac{x}{2D}\sin 2x. \qquad (PI)_2 = \frac{1}{-5}\cos 3x.$

$$= \frac{-x}{4}\cos 2x$$

$\circledast$ $(D-2)^3 y = x e^{2x}$

$D = 2, 2, 2$

$CF = (c_1 + c_2 x + c_3 x^2) e^{2x}$

$$PI = \frac{1}{(D-2)^3} x e^{2x}$$

$$= e^{2x} \frac{1}{((D+2)-2)^3} x$$

$$= e^{2x} \frac{1}{D^3} x$$

$$= e^{2x} \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{x^4}{4}$$

$$y = (c_1 + c_2 x + c_3 x^2) e^{2x} + \frac{x^4 e^{2x}}{24}$$

$\circledast$ $y'' - 2y' + y = x - 1$

$(D^2 - 2D + 1) = 0$

$(D-1)^2 = 0 \qquad D = 1, 1$

$CF = (c_1 + c_2 x) e^x$

$$PI = \frac{1}{(D^2 - 2D + 1)} (x-1)$$

$$= (1 + D^2 - 2D)^{-1} (x-1)$$

$$= (1 - (D^2 - 2D)) (x-1)$$

$$PI = (x-1) + 2 = x + 1$$

$\circledast$ $\dfrac{d^2 y}{dx^2} = e^x \cos x$

$$PI = \frac{1}{D^2} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2} \cos x$$

$$= e^x \left[\frac{1}{D^2+2D+1}\right] \cos x = \frac{e^x}{2} \sin x$$

$$\frac{d^2y}{dx^2} + y = x ; \quad \text{at } x = 0 , y = 1$$

$$\& \quad x = \pi/2 , y = \pi/2$$

$$D^2 + 1 = 0$$

$$D = \pm i$$

$$CF = (c_1 \cos x + c_2 \sin x)$$

$$PI = \frac{1}{D^2+1} x$$

$$= (1 + D^2)^{-1} x$$

$$= (1 - D^2) x \Rightarrow x - 0 = x$$

$$y = c_1 \cos x + c_2 \sin x + x$$

$$x = 0 , y = 1 \Rightarrow 1 = c_1$$

$$x = \pi/2 , y = \pi/2 \Rightarrow \frac{\pi}{2} = c_2 + \pi/2 \Rightarrow c_2 = 0$$

$$y = \cos x + x$$

$$y'' - 3y' + 2y = \cosh x$$

$$= \frac{e^x + e^{-x}}{2}$$

$$D^2 - 3D + 2 = 0 \Rightarrow D^2 - 2D - D + 2 = 0$$

$$D = 1, 2$$

$$CF = c_1 e^x + c_2 e^{2x}$$

$$PI = \frac{1}{2}\left[\frac{1}{\quad}\right] e^x + \frac{1}{2}\left[\frac{1}{\quad}\right] e^{-x}$$

(*) $y'' - 3y' + 2y = e^x \;;\quad y = 3 \;\;\&\;\; \frac{dy}{dx} = 3$ at $x = 0$.

$$D^2 - 3D + 2 = 0$$

$$D = 1, 2$$

$$CF = c_1 e^x + c_2 e^{2x}$$

$$PI = \left[\frac{1}{D^2 - 3D + 2}\right] e^x$$

$$= -x e^x$$

$$y = c_1 e^x + c_2 e^{2x} - x e^x$$

$$y' = c_1 e^x + 2 c_2 e^{2x} - e^x (x+1)$$

$x = 0,\ y = 3 \Rightarrow 3 = c_1 + c_2$

$x = 0,\ y' = 3 \Rightarrow 3 = c_1 + 2c_2 - 1$

$$c_2 - 1 = 0$$

$$c_2 = 1 \qquad c_1 = 2$$

$\Rightarrow$ Cauchy's homogenous D.eq^n :-

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1} x \frac{dy}{dx} + k_n y = X$$

$k_1, k_2 \ldots$ are constants.

$x = e^t$ X is function of x.

it reduces to linear DE with constant coef.

$$x \frac{dy}{dx} = D_1 y = \frac{dy}{dt}$$

$$x^2 \frac{d^2 y}{dx^2} = D_1 (D_1 - 1) y$$

$$x^3 \frac{d^3 y}{dx^3} = D_1 (D_1 - 1)(D_1 - 2) y$$

$$\cancel{(D_1^2 + D_1 + 1)y = 4\cos t}$$

$$\cancel{PI = \left[\frac{1}{D_1^2 + D_1 + 1}\right] 4\cos t}$$

$$\cancel{= 4\left(\frac{1}{D_1}\cos t\right)}$$

$$\cancel{= 4\sin t}$$

$$(D_1(D_1 - 1) + D_1 + 1)y = 4\cos t$$

$$(D_1^2 + 1)y = 4\cos t$$

$$PI = 4\left(\frac{1}{D_1^2+1}\right)\cancel{4}\cos t$$

$$= 2t\sin t$$

$$= 2\log(1+x)\sin[\log(1+x)]$$

⇒ Method of variation of parameters :-

This is a method to find the PI of

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R}$$

1. $CF = c_1 y_1 + c_2 y_2$
2. $PI = Ay_1 + By_2$

$$A = -\int \frac{Ry_2}{\omega}\,dx, \quad B = \int \frac{Ry_1}{\omega}\,dx$$

$$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

3. $y = CF + PI$

Note :- If P & Q are functions of x then we should know two independent soln y_1, y_2 of the corresponding homogenous eqn $y'' + Py' + Qy = 0$ so that we can write CF.
$(CF = c_1 y_1 + c_2 y_2)$.

$$y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$$

$$D^2 - 6D + 9 = 0$$

$$D = 3, 3$$

$$CF = c_1 e^{3x} + c_2 x e^{3x}$$

$$y_1 = e^{3x} \quad y_2 = x e^{3x}$$

$$w = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix}$$

$$= e^{6x} + 3xe^{6x} - 3xe^{6x}$$

$$w = e^{6x}$$

$$A = -\int \frac{\frac{\cancel{e^{3x}}}{x^2} \cdot x\cancel{e^{3x}}}{\cancel{e^{6x}}} dx = -\int \frac{1}{x} = -\log x.$$

$$B = +\int \frac{\cancel{e^{3x}}}{x^2} \cdot \frac{\cancel{e^{3x}}}{\cancel{e^{6x}}} dx = \frac{-1}{x}.$$

$$PI = -\log x \; e^{3x} - \frac{1}{x} x e^{3x}$$

$$= -e^{3x}(\log x + 1).$$

The PI of $\frac{d^2y}{dx^2} + 4y = \sec^2 2x$ is $Ay_1 + By_2$

where $y_1 = \cos 2x$, $y_2 = \sin 2x$ find A & B

$x=0$

(*) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 0.$

Sub. $x = e^t \Rightarrow \log x = t$

$(D_1(D_1-1) - 3D_1 + 4)y = 0.$

$(D_1^2 - 4D_1 + 4)y = 0.$

$D_1 = 2, 2$

$y = (c_1 + c_2 t)e^{2t}.$

$y = (c_1 + c_2 \log x)x^2$ //

(*) Find PI of $x^2y'' - xy' + 3y = x^2 \log x.$

Sub. $x = e^t \Rightarrow \log x = t$

$(D_1(D_1-1) - D_1 + 3)y = e^{2t}.t$

$(D_1^2 - 2D_1 + 3)y = te^{2t}.$

$D_1 = -1, 3$

$D^2 - 3D + D + 3$

$D(D-3)$

$$PI = \left[\frac{1}{D_1^2 - 2D_1 + 3}\right] t e^{2t}$$

$= x$

$$= e^{2t}\left[\frac{1}{(D_1+2)^2 - 2(D_1+2) + 3}\right] t$$

$$= e^{2t}\left[\frac{1}{D_1^2 + 2D_1 + 3}\right] t$$

coef.

$$= \frac{e^{2t}}{3}\left[1 - \left(\frac{D_1^2}{3} + \frac{2D_1}{3}\right)\right] t$$

$$= \frac{e^{2t}}{3}\left[t - \frac{2}{3}\right].$$

(*) $$PI = \frac{x^2}{3}\left[\log x - \frac{2}{3}\right]$$

Legenders eq^n :-

$$\Rightarrow (a+bx)^n \frac{d^n y}{dx^n} + K_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots K_{n-1} (a+bx) \frac{dy}{dx} + K_n y = X.$$

$(a+bx) = e^t \downarrow$

Linear DE with const coef.

$$(a+bx) \frac{dy}{dx} = b D_1 y \qquad \boxed{t = \log(a+bx)}$$

$$(a+bx)^2 \frac{d^2y}{dx^2} = b^2 D_1 (D_1 - 1) y$$

$$(a+bx)^3 \frac{d^3y}{dx^3} = b^3 D_1 (D_1 - 1)(D_1 - 2) y.$$

$$D_1 = \frac{d}{dt}, \quad D_1^2 = \frac{d}{dt^2}, \ldots$$

$$(1+2x)^2 y'' - 6(1+2x) y' + 16y = 0$$

$$4 D_1 (D_1 - 1) - 6(2) D_1 + 16 = 0$$

$$4D_1^2 - 8D_1 + 16 = 0 \qquad (1+2x) = e^t.$$

$$D_1^2 - 2D_1 + 4 = 0$$

$$D_1 = 2, 2$$

$$CF = (c_1 + c_2 t) e^{2t}$$

$$\cancel{PI} = \cancel{\left[\frac{1}{D_1^2 - 2D_1 + 4}\right]}$$

$$CF = \left\{c_1 + c_2 \log(1+2x)\right\} (1+2x)^2$$

PI of $(1+x)^2 y'' + (1+x) y' + y = 4 \cos(\log(1+x))$

$$(1+x) = e^t$$

* Res. of $f(z) = \frac{1}{(z^2+1)^2}$ @ $z = i$ is

$f(z) = \frac{1}{(z+i)^2(z-i)^2} \rightarrow \lim_{z\to i} \frac{d}{dz}\left\{ \cancel{(z-i)^2} \times \frac{1}{(z+i)^2\cancel{(z-i)^2}} \right\}$

$\lim_{z\to i} \frac{-2}{(z+i)^3} \Rightarrow \frac{-2}{(2i)^3} = \frac{-2}{8i^3} = \frac{1}{4i}$

* $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ Calculate res. at ech of the poles.

$z = 1, \quad z = -2$

Res at $z=1 \rightarrow \lim_{z\to 1} \frac{d}{dz}\left\{ \cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2}(z+2)} \right]$

$\Rightarrow \lim_{z\to 1} \frac{d}{dz}\left[\frac{z^2}{z+2} \right] \Rightarrow \lim_{z\to 1} \left[\frac{2z}{z+2} - \frac{z^2}{(z+2)^2} \right]$

$= \frac{2}{3} - \frac{1}{9} = \frac{6-1}{9} = \frac{5}{9}$

Res at $z=-2 \rightarrow \lim_{z\to -2} \cancel{(z+2)} \frac{z^2}{(z-1)^2\cancel{(z+2)}} \Rightarrow \frac{4}{9}$

* Res. of $f(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles

a) $1/2, -1/2, 1$ b) $1/2, 1/2, -1$ c) $1/2, 1, -3/2$ d) $1/2, -1, 3/2$

At $z \Rightarrow 0$ $\lim_{z\to 0} \frac{1-2z}{(z-1)(z-2)} = \frac{1}{2}$

At $z \Rightarrow 1$ $\lim_{z\to 1} \frac{1-2z}{z(z-2)} = \frac{-1}{-1} = 1$

At $z \Rightarrow 2$ $\lim_{z\to 2} \frac{1-2z}{z(z-1)} = \frac{-3}{2}$

⊛ $f(z) = \dfrac{1}{(z+2)^2(z-2)^2}$ at $z = 2$ is

$$\lim_{z\to 2} \frac{d}{dz}\left\{ \cancel{(z-2)^2} \times \frac{1}{(z+2)^2\cancel{(z-2)^2}} \right\} \to \lim_{z\to 2} \frac{-2}{(z+2)^3} = \frac{-2}{64}$$

$$= \frac{-1}{32}$$

Cauchys' Residue theorem.

$f(z)$ is analytic in a closed curve c except at a finite no. of points, then $\int_c f(z)\,dz = 2\pi i \{$ sum of residues $f(z)$ at each of poles

⊛ $\oint_c \dfrac{1}{(z+2)^2(z-2)^2}\,dz \qquad |z| = 5$

$$= 2\pi i \left[\text{Res}(2) + \text{Res}(-2) \right]$$

$$= 2\pi i \left[\lim_{z\to 2} \frac{d}{dz} \frac{1}{(z+2)^2} + \lim_{z\to -2} \frac{d}{dz} \frac{1}{(z-2)^2} \right]$$

$$= 2\pi i \left[\lim_{z\to 2} \frac{-2}{(z+2)^3} + \lim_{z\to -2} \frac{-2}{(z-2)^3} \right] = 2\pi i \left[\frac{-2}{64} + \frac{2}{64} \right]$$

$$= \underline{\underline{0}}$$

⊛ $\oint_c \dfrac{e^z}{z^2+1}\,dz$ where c is the region $|z| = 2$

$$\oint_c \frac{e^z}{(z-i)(z+i)} = 2\pi i \{ \text{Res}(i) + \text{Res}(-i) \}$$

$$= 2\pi i \left\{ \lim_{z\to i} \frac{e^z}{z+i} + \lim_{z\to -i} \frac{e^z}{z-i} \right\}$$

$$= 2\pi i \left\{ \frac{e^i}{2i} + \frac{e^{-i}}{-2i} \right] = \pi \left[e^i - e^{-i} \right]$$

⊛ $\int_c \dfrac{z^2+1}{z(2z+1)}\,dz \qquad |z| = 1$

$$\frac{1}{2} \int_c \frac{z^2+1}{z(z+\frac{1}{2})} = \frac{2\pi i}{2} \{ \text{Res}(0) + \text{Res}(-1/2) \}$$

$$= \pi i \left\{ \lim_{z\to 0} \frac{z^2+1}{z+1/2} + \lim_{z\to -1/2} \frac{z^2+1}{z} \right\}$$

$-\frac{5}{2}$

$$= \pi i \left\{ 2 + -\frac{5}{2} \right] = \underline{\underline{\frac{-\pi i}{2}}}$$